MATH 245 S19, Exam 2 Solutions

1. Carefully define the following terms: Nonconstructive Existence theorem, Proof by Shifted Induction, Proof by Strong Induction.

   The Nonconstructive Existence Theorem states that if $\forall x \in D, \neg P(x)$ is a contradiction, then $\exists x \in D, P(x)$ is true. Let $s \in \mathbb{Z}$. To prove the proposition $\forall x \in \mathbb{Z}$ with $x \geq s$, $P(x)$ by shifted induction, we must (a) Prove that $P(s)$ is true; and (b) Prove that $\forall x \in \mathbb{Z}$ with $x \geq s$, $P(x) \rightarrow P(x+1)$. To prove the proposition $\forall x \in \mathbb{N}$, $P(x)$ by strong induction, we must (a) Prove that $P(1)$ is true; and (b) Prove that $\forall x \in \mathbb{N}$, $P(1) \wedge P(2) \wedge \cdots \wedge P(x) \rightarrow P(x+1)$.

2. Carefully define the following terms: recurrence, big Omega, big Theta.

   A recurrence is a sequence with the property that all but finitely many of its terms are defined in terms of its previous terms. Let a_n, b_n be sequences. We say that a_n is big Omega of b_n if $\exists n_0 \in \mathbb{N}$, $\exists M \in \mathbb{R}$, $\forall n \geq n_0$, $M|a_n| \geq |b_n|$ holds. Let a_n, b_n be sequences. We say that a_n is big Theta of b_n if a_n is big O of b_n and also a_n is big Omega of b_n.

3. Let $a, b \in \mathbb{Z}$ with $b \geq 1$. Use minimum element induction to prove $\exists q, r \in \mathbb{Z}$ with $a = bq + r$ and $0 < r \leq b$.

   Let $S = \{m \in \mathbb{Z} : m \geq \frac{a}{b} - 1\}$, which is a nonempty set of integers. It has lower bound $\frac{a}{b} - 1$, so by minimum element induction it must have a minimum element, which we call q. Since $q \in S$, we have $q \in \mathbb{Z}$ and $q \geq \frac{a}{b} - 1$. Hence $bq \geq a - b$, which rearranges to $b \geq a - bq$. Set $r = a - bq$; by the above calculation $b \geq r$. Since q was minimal in S, $q - 1 \notin S$. Since $q \in \mathbb{Z}$ we must have $q - 1 < \frac{a}{b} - 1$, or $q < \frac{a}{b}$. We have $qb < a$, which rearranges to $0 < a - bq = r$. Combining, we have $0 < r \leq b$.

4. Let $x \in \mathbb{R}$. Prove that $\lceil x \rceil$ is unique; i.e., prove there is at most one $n \in \mathbb{Z}$ with $n - 1 < x \leq n$.

   Suppose there were two integers n, n', satisfying $n - 1 < x \leq n$ and also $n' - 1 < x \leq n'$. Combining $n - 1 < x$ with $x \leq n'$, we get $n - 1 < n'$. Combining $n' < x + 1$ with $x + 1 \leq n + 1$, we get $n' < n + 1$. Hence, we have $n - 1 < n' < n + 1$. By a theorem from the book (1.12d), we must have $n = n'$.

5. Let F_n denote the Fibonacci numbers. Prove that for all $n \in \mathbb{N}$, we have $F_{2n} = \displaystyle\sum_{i=0}^{n-1} F_{2i+1}$.

   We prove by (ordinary) induction. The base case is $n = 1$: we have $F_{2\cdot 1} = F_2 = 1$, while the sum has just one term, namely $F_{2\cdot 0+1} = F_1 = 1$. Now, let $n \in \mathbb{N}$ be arbitrary, and assume that $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$. We add F_{2n+1} to both sides, getting $F_{2n+1} + F_{2n} = F_{2n+1} + \sum_{i=0}^{n-1} F_{2i+1}$. Now, $F_{2n+1} + F_{2n} = F_{2n+2} = F_{2(n+1)}$ by the Fibonacci recurrence (since $2n + 2 \geq 2$). Also, $F_{2n+1} + \sum_{i=0}^{n-1} F_{2i+1} = \sum_{i=0}^{n} F_{2i+1}$. Combining, we get $F_{2(n+1)} = \sum_{i=0}^{n} F_{2i+1}$.

6. Prove that for all $n \in \mathbb{N}$ with $n \geq 4$, we have $n! > 2^n$.

We prove by (shifted) induction. The base case is $n = 4$: we have $4! = 24 > 16 = 2^4$. Now, let $n \in \mathbb{N}$ with $n \geq 4$, and assume that $n! > 2^n$. We multiply both sides by $n+1$, getting $(n+1)n! > 2^n(n+1)$. Now, $(n+1)n! = (n+1)!$ by the factorial definition, since $n+1 \geq 1$. Also, $n+1 > 2$ (since $n \geq 4$), so $2^n(n+1) > 2^n \cdot 2 = 2^{n+1}$. Combining, we get $(n+1)! > 2^{n+1}$.

7. Let $a_n = n^{1.9} + n^2$. Prove that $a_n = O(n^2)$.

Set $n_0 = 1$ and $M = 2$. Let $n \geq n_0 = 1$ be arbitrary. We have $n^{0.1} \geq 1 = n^0$; multiplying both sides by the positive $n^{1.9}$ we get $n^2 \geq n^{1.9}$. Hence $|a_n| = a_n = n^{1.9} + n^2 \leq n^2 + n^2 = 2n^2 = 2|n^2| = M|n^2|$.

8. Solve the recurrence given by $a_0 = 2, a_1 = 6, a_n = 5a_{n-1} - 6a_{n-2}$ $(n \geq 2)$.

Our characteristic polynomial is $r^2 - 5r + 6 = (r-2)(r-3)$. It has two distinct roots, $2, 3$. Hence, the general solution to the recurrence is $a_n = A2^n + B3^n$. We now apply the initial conditions. $2 = a_0 = A2^0 + B3^0 = A + B$. $6 = a_1 = A2^1 + B3^1 = 2A + 3B$. We solve the system $\{A + B = 2, 2A + 3B = 6\}$ to find $B = 2, A = 0$. Hence, the specific solution to the recurrence is $a_n = 2 \cdot 3^n$.

9. Prove that for all $x \in \mathbb{R}$, we have $|x-1| + |x+2| \geq 3$.

Let $x \in \mathbb{R}$ be arbitrary. We have three cases, depending on x: $(-\infty, -2), [-2, 1), [1, +\infty)$:
Case $x < -2$: We have $|x-1| + |x+2| = -(x-1) - (x+2) = -2x - 1$. Since $x < -2$, we multiply by -2 to get $-2x > (-2)(-2) = 4$, so $-2x - 1 > 4 - 1 = 3$.
Case $-2 \leq x < 1$: We have $|x-1| + |x+2| = -(x-1) + (x+2) = 3$. This is certainly ≥ 3.
Case $1 \leq x$: We have $|x-1| + |x+2| = (x-1) + (x+2) = 2x + 1$. Since $x \geq 1$, we multiply by 2 to get $2x \geq 2$ and hence $2x + 1 \geq 2 + 1 = 3$.

10. Let $x \in \mathbb{R}$. Prove that $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor$ if and only if $x - \lfloor x \rfloor < \frac{1}{2}$.

Note: "If and only if" means there are two things to prove.

SOLUTION 1: Suppose first that $x - \lfloor x \rfloor < \frac{1}{2}$. We add $\frac{1}{2}$ to both sides and re arrange to get $x + \frac{1}{2} < \lfloor x \rfloor + 1$. But also $x + \frac{1}{2} > x \geq \lfloor x \rfloor$. Hence $\lfloor x \rfloor \leq x + \frac{1}{2} < \lfloor x \rfloor + 1$. Hence $\lfloor x \rfloor$ and $\lfloor x + \frac{1}{2} \rfloor$ are both integers that satisfy the same two inequalities; by the uniqueness of $\lfloor x + \frac{1}{2} \rfloor$, they must be equal.

Suppose now that $x - \lfloor x \rfloor \geq \frac{1}{2}$. We add $\frac{1}{2}$ to both sides and rearrange to get $x + \frac{1}{2} \geq \lfloor x \rfloor + 1$. By a theorem from the book (5.16a), we have $\lfloor x + \frac{1}{2} \rfloor \geq \lfloor \lfloor x \rfloor + 1 \rfloor$. By another theorem from the book (5.17a), we have $\lfloor \lfloor x \rfloor + 1 \rfloor = \lfloor x \rfloor + \lfloor 1 \rfloor = \lfloor x \rfloor + 1$. Combining, $\lfloor x + \frac{1}{2} \rfloor \geq \lfloor x \rfloor + 1$; in particular, $\lfloor x + \frac{1}{2} \rfloor \neq \lfloor x \rfloor$.

SOLUTION 2: Note that since $\lfloor x \rfloor$ is an integer, by a theorem from the book (5.17a), $\lfloor x + \frac{1}{2} \rfloor - \lfloor x \rfloor = \lfloor (x - \lfloor x \rfloor) + \frac{1}{2} \rfloor = \lfloor A \rfloor$. Now, if $x - \lfloor x \rfloor < \frac{1}{2}$, then $A < 1$, so $\lfloor A \rfloor \leq 0$ and hence $\lfloor x + \frac{1}{2} \rfloor \leq \lfloor x \rfloor$. But also $\lfloor x + \frac{1}{2} \rfloor \geq \lfloor x \rfloor$ (by a theorem from the book, 5.16a, since $x + \frac{1}{2} \geq x$), so $\lfloor x + \frac{1}{2} \rfloor = \lfloor x \rfloor$. If instead $x - \lfloor x \rfloor \geq \frac{1}{2}$, then $1 \leq A$, so $\lfloor A \rfloor \geq 1$ and hence $\lfloor x + \frac{1}{2} \rfloor \neq \lfloor x \rfloor$.